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# ITERATIVE DETERMINATION OF HOMOCLINIC ORBIT PARAMETERS AND PADÉ APPROXIMANTS

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## 1. INTRODUCTION

Iterative procedures are often used in the determination of the parameters of homoclinic orbits exhibited by dynamical systems either using numerical [1–7] or analytical computations [8]. The aim of this letter is to present a more economical and simplified approach. As it is already known, Padé approximants can be applied to improve the rate of convergence of iteration procedures [9, 10]. This question will be discussed further in this letter.

## 2. RESULTS OF ITERATIVE TECHNIQUE

In order to illustrate the application of Padé approximants, the iterative technique results obtained in reference [8] will be considered. In that reference the iterative technique is used to obtain an analytical approximation to the homoclinic loops of the Lorenz system

$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} = \delta(y - x),$$

$$\dot{y} \equiv \frac{\mathrm{d}y}{\mathrm{d}t} = \rho x - y - zx, \qquad (1)$$

$$\dot{z} \equiv \frac{\mathrm{d}z}{\mathrm{d}t} = xy - bz,$$

for the following fixed parameter values $\delta = 10$, $b = \frac{8}{3}$ and $\rho = 13{\cdot}926\ldots$. First, the local structure of the homoclinic solution for $t \to \pm 0$ and $\pm\infty$ was analyzed and then the global solutions were constructed on the basis of quasifractional approximants [9, 10]. The accuracy of the approximation is improved iteratively. Since the Lorenz system is autonomous, one can arbitrarily choose the initial conditions of the orbit to be the

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TABLE 1

*Iterations and results*

| No. of iterations | $y_0$ | $z_0$ | $\rho$ |
|---|---|---|---|
| 0 | 15·08854 | 28·92802 | 17·63087 |
| 1 | 11·55446 | 19·02569 | 13·45975 |
| 2 | 11·65734 | 19·27001 | 13·91258 |

point where $x(0) = y(0) = y_0$ and $z(0) = z_0$. Fixing two of the Lorenz parameters to the values $\gamma = 10$, $b = \frac{8}{3}$, each iteration provides estimates for the initial conditions for the homoclinic orbit and the value $\rho$. The results for three successive iterations are given in Table 1.

The accuracy control is related to the values of the parameter $\rho$.

## 3. ACCELERATION OF ITERATION PROCEDURE CONVERGENCE USING PADÉ APPROXIMANTS

A succession of Padé approximants is mainly dependent on higher order approximations of an asymptotic process. This principal difficulty can be sometimes overcome by using symbolic computations, but in general this problem seems to be open [10]. More suitably oriented approaches to eliminate this difficulty are referred to iteration procedures [10, 11].

Consider the following iteration procedure:

$$T(u_0) = 0, \quad u_n = T(u_{n-1}), \quad u = 1, 2, 3, \ldots$$

where $T$ is a certain operator. Let us introduce a formal small parameter $\varepsilon$ ($0 \leqslant \varepsilon \leqslant 1$) and a solution $u \approx u_n$ is sought in powers of $\varepsilon$, of the form

$$u \approx u_0 + (u_1 - u_0)\varepsilon + (u_2 - u_1)\varepsilon^2 + \cdots + (u_n - u_{n-1})\varepsilon^n. \tag{2}$$

For $\varepsilon = 0$ one has $u = u_0$, whereas for $\varepsilon = 1$ one gets $u \approx u_n$. The series (2) can be approximated by a rational function due to the Padé scheme

$$\frac{u_0 + \sum_{i=1}^{m} a_i \varepsilon^i}{1 + \sum_{j=1}^{l} \beta_j \varepsilon^j} - [u_0 + (u_l - u_0)\varepsilon + \cdots + (u_n - u_{n-1})\varepsilon^n] = 0(\varepsilon^{n+1}), \tag{3}$$

where $m + l = n$.

Therefore, for $\varepsilon = 1$ one obtains

$$u \approx \frac{u_0 + \sum_{i=1}^{m} a_i}{1 + \sum_{j=1}^{l} b_j}. \tag{4}$$

For $m = 1$, a diagonal Padé approximant is obtained. Various examples (see references [10, 11] and references cited therein) show the high efficiency of the described approach.

TABLE 2

*Results obtained using Padé approximants*

| $y_0$ | $z_0$ | $\rho$ |
|---|---|---|
| 11·65734 | 19·27001 | 13·9283 |

## 4. ACCURACY IMPROVEMENT OF THE LORENZ HOMOCLINIC ORBIT PARAMETERS DETERMINATION

The calculations indicate that the use of a zero order approximation for the iterational technique leads to relatively high errors. Therefore, it seems to be more efficient to begin with the first approximation and to use the equation

$$u \approx u_1 + (u_2 - u_1)\varepsilon.$$

The corresponding Padé approximant for $\varepsilon = 1$ gives the result

$$u \approx \frac{u_1^2}{2u_1 - u_2}.$$

The computational results are included in Table 2.

Following the methods of reference [8], one can estimate the accuracy of the calculated parameter $\rho$. Hassard and Zhang [2] used numerical shooting to provide improved bounds on the value $\rho$ for which the homoclinic orbit exists. They showed that for $\gamma = 10$ and $b = \frac{8}{3}$, $\rho \in [13{\cdot}9265, 13{\cdot}9270]$.

Accordingly, $\rho$ in Table 2 is obtained with an error less than 0·013%, whereas the third iteration has an error of 0·096%. As it can be seen, without any additional calculations the accuracy of getting initial conditions for homoclinic orbits of the Lorenz system is increased by one order of magnitude.

The authors have checked the method earlier using a more complex system governing the vocal cords oscillations [12].

To compute the whole picture of the vocal cords bifurcation diagram (note that there are now three second order differential equations) one needs 34% less time than when proceeding with a usual path-following method, which using a PC, the IMSL subroutines and Fortran needs 40 min of computations. Depending on the followed bifurcation branch sometimes one needs to omit three or even four iterations holding the required accuracy, which with respect to the norm was equal to $10^{-6}$. Avoiding each of a few iterates means that one integrates numerically a few time less the investigated ODEs in a whole interval of the period of a periodic solution being investigated. Now, especially if the equations are stiff or they need a special numerical treatment the advantages of the Padé approximants are very high.

## 5. CONCLUDING REMARKS

We conclude with two important steps. The first one relates to the observation that the Padé approximants for $m-1(P[m-1])$, $m(P[m])$ and $m+1(P[m+1])$ satisfy the following inequality:

$$P[m-1] \leqslant P[m] \leqslant P[m+1].$$

This means that the intervals for the change of a sought parameter can be estimated. The second is related to the observation described in references [10, 13, 14]. A convergence between an iterative procedure and the Padé approximants essentially increases the heuristic reliability of the obtained results and can help in their theoretical implementations. Thus, the Padé approximants can be used not only for the construction of the homoclinic orbits, but also to improve the accuracy of both the analytical and numerical iterational procedures.

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